Because the function $\log(\cdot)$ is a monotonically increasing function, we can form a different criterion that contains exactly the same information as $J_{\mathrm{MDL}}(p')$. The new criterion chosen is

$$J(p') = \left(\prod_{i=1}^{m} \varepsilon_i\right) m(k+1)^{mp'/(k+1)} \tag{14}$$

There are several ways to choose the instruments and the matrix Z_k. In this Note we consider only the special case where $z_t = y_{t-h}$, and for the selection of instruments we consider different values of h. We can then form the new multivariate minimum description length criterion (14) and search for an abrupt change in this criterion for different values of p'.

Experimental Results

The procedure for multivariate AR-order determination, or for the estimation of the number of modes in a frequency band, is now applied in the laboratory to a real structure. In this experience we consider a U beam with two accelerometers (Fig. 1). This beam is randomly excited. The signals are sampled at the rate $\Delta t = 625\ \mu$s, and 16,384 points are collected for each channel.

Our purpose is to determine the number of modes in the frequency band [0; 800 Hz] from output accelerometers only. Figures 2 and 3 show the frequency response of accelerometers. From these figures it is very difficult to determine the number of modes of this beam in the frequency band considered by counting the number of peaks of resonance, the number of apparent peaks and very weak peaks varying between 5 and 8.

These figures do not allow us to know the exact number of modes in the frequency band of interest. To resolve this problem, we use the $J(p')$ criterion. For the selection of the instrumental variables $z_t = y_{t-h}$, we have considered different values of h and found that all of these selections could give the satisfactory results. Table 1 shows the statistical results of the criterion $J(p')$ when the number of block rows in the matrix of instruments Z_k changes. In these experimental results we use the instrumental variable $z_t = y_{t-1}$. The true order p of the AR part of the multivariate ARMA process is obtained by inspection of the values of $J(p')$: when a break occurs, we determine the order of the AR part.

By inspection of Table 1, the true order of the AR part is $p = 6$, and the number of modes in the frequency band considered is $n = 6$.

Conclusions

A time-domain procedure for the determination of the number of modes in a frequency band, from the time response delivered by the output of accelerometers only, has been proposed. Based on a combination of the multivariate minimum description length and the overdetermined instrumental variable scheme, an efficient method for AR-order determination of a multivariate ARMA model has been developed. Experimental results have shown the effectiveness of our method. It may be interesting to study other instrumental variable selections and to generalize this method to large industrial structures at work.

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Correcting System Matrices Using the Orthogonality Conditions of Distinct Measured Modes

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Nomenclature

$[K]$	= actual stiffness matrix
$[K_o]$	= analytical stiffness matrix
$[M]$	= actual mass matrix
$[M_o]$	= analytical mass matrix
$\boldsymbol{m}_{\mathrm{actual}}$	= vector of actual masses
$\boldsymbol{m}_{\mathrm{analytical}}$	= vector of analytical masses
$\boldsymbol{m}_{\mathrm{update}}$	= vector of updated masses
N	= degrees of freedom of the analytical model
N_e	= number of measured modes
$\boldsymbol{x}_j$	= jth measured mode shape
$(x_i)_r$	= rth element of $\boldsymbol{x}_i$
$\delta\boldsymbol{k}$	= vector of stiffness corrections
$[\delta K]$	= correction stiffness matrix
$[\delta M]$	= correction mass matrix
$\delta\boldsymbol{m}$	= vector of mass corrections
δm_{rs}	= (r, s)th element of $[\delta M]$
ϵ_k	= error parameter for the updated stiffnesses
ϵ_m	= error parameter for the updated masses
ϵ_λ	= error parameter for the updated eigenvalues
$(\lambda'_j, \boldsymbol{x}'_j)$	= jth mode of vibration of the mass-modified updated system
$(\lambda''_j, \boldsymbol{x}''_j)$	= jth mode of vibration of the mass-modified actual system

Introduction

WITH the arrival of digital computers, new methods of analysis have been developed to analyze and predict the dynamical behavior of complex structures, especially in the method of finite elements. Once the finite element model of a physical system is constructed, it is often validated by comparing its analytical modes of vibration with the results of a modal survey. If the agreement between the two is good, then the analytical model can be used with confidence for future analysis. If the correlation between the two is poor, then assuming the measured data to be exact, the finite element model must be corrected or updated such that the correlation between analytical predictions and test data is improved. Many model updating schemes have been developed over the years to adjust the analytical finite element models using test data. Detailed discussion of every approach is beyond the scope of this Note, and interested readers are referred to the recent survey paper by Mottershead and Friswell.[1] In this Technical Note, new model updating schemes to adjust the system mass and stiffness matrices are developed.

Proposed Model Updating Algorithms

Like the perturbation model updating approach introduced in Ref. 2, the orthogonality constraints will also be used in this Note to update the system matrices. The proposed schemes, however, are

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based on enforcing the orthogonality conditions of distinct measured mode shapes with respect to the mass and stiffness matrices of the actual system. The new algorithms offer numerous advantages over the updating scheme introduced in Ref. 2. First, because no assumptions are made regarding the magnitudes of the correction matrices relative to the analytical matrices, the proposed updating approaches can be applied even when the analytical and the actual system matrices deviate substantially from each other. Second, because only the orthogonality conditions of distinct measured mode shapes are utilized in the updating procedure, the measured eigenvectors do not need to be normalized in any specific manner. Finally, by use of the proposed updating schemes, the connectivity information of the mass and stiffness matrices can be easily imposed. Thus, the updated matrices preserve the physical configuration of the system.

Because of the orthogonality condition of the mode shapes, the measured eigenvectors corresponding to distinct eigenvalues must obey $\boldsymbol{x}_i^T[M]\boldsymbol{x}_j = 0$, where $i \neq j$. Substituting $[M] = [M_o] + [\delta M]$ into the preceding equation, we get

$$\boldsymbol{x}_i^T[\delta M]\boldsymbol{x}_j = \sum_{r=1}^{N}\sum_{s=1}^{N}\delta m_{rs}(x_i)_r(x_j)_s = -\boldsymbol{x}_i^T[M_o]\boldsymbol{x}_j \qquad (1)$$

For $i, j = 1, \ldots, N_e$ and $i \neq j$, Eq. (1) leads to a set of $(N_e^2 - N_e)$ equations that can be expressed conveniently in compact matrix form as $[A]\boldsymbol{\delta m} = \boldsymbol{r}$, where the unknown column vector $\boldsymbol{\delta m}$ is given by $\boldsymbol{\delta m} = [\delta m_{11} \cdots \delta m_{1N} \mid \cdots \mid \delta m_{N1} \cdots \delta m_{NN}]^T$. Incidentally, note that, because $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ appear on both sides of Eq. (1), any arbitrary constants in the measured modes will not affect the resulting $(N_e^2 - N_e)$ equations.

Once the mass matrix has been updated, the stiffness matrix can be corrected or adjusted by having it satisfy $[K][X] = [M][X][\Lambda]$, that is, the generalized eigenvalue problem associated with the free response of the structure. Premultiplying the generalized eigenvalue problem by $[X]^T$, setting $[K] = [K_o] + [\delta K]$, and rearranging, we get

$$[X]^T[\delta K][X] = [X]^T[M][X][\Lambda] - [X]^T[K_o][X] \qquad (2)$$

Like before, this matrix equation can be manipulated into $[B]\boldsymbol{\delta k} = \boldsymbol{h}$.

Both $[A]\boldsymbol{\delta m} = \boldsymbol{r}$ and $[B]\boldsymbol{\delta k} = \boldsymbol{h}$ are of the form $[G]\boldsymbol{y} = \boldsymbol{b}$, where matrix $[G]$ and vector $\boldsymbol{b}$ are both known and of size $m \times n$ and length m, respectively, and $\boldsymbol{y}$ is a vector of length n. Initially it appears that two least-squares problems of size $m \times n$ need to be solved to update the system mass and stiffness matrices. However, having manipulated the correction matrices into vector forms, the optimal matrix storage scheme commonly used in finite elements[3] can be applied to impose the connectivity information by eliminating all of the known zero elements from $\boldsymbol{y}$ and by deleting all of the corresponding columns in $[G]$. Thus, the physical configuration of the system is preserved, and the size of the numerical problems that need to be solved is drastically reduced. For example, if the mass matrix is diagonal, then $\delta m_{ij} = 0$ for $i \neq j$, and $[A]\boldsymbol{\delta m} = \boldsymbol{r}$ reduces to $[A']\boldsymbol{\delta m'} = \boldsymbol{r}$, where $[A']$ is obtained from $[A]$ by deleting all of the columns that multiply by δm_{ij} for $i \neq j$ and $\boldsymbol{\delta m'} = [\delta m_{11} \;\; \delta m_{22} \cdots \delta m_{NN}]^T$. Thus, the initial problem of size $(N_e^2 - N_e) \times N^2$ is reduced to one of size $(N_e^2 - N_e) \times N$. Similarly, the connectivity information of the stiffness matrix can be enforced to preserve the load paths and to reduce the size of the least-squares problem to be solved. In the subsequent numerical experiments, the CMLIB routine sglss will be accessed to solve the least-squares problems.

Results

Consider the system of Fig. 1 with 25 oscillators, whose mass matrix is diagonal and whose stiffness matrix is symmetric and tridiagonal. For the purpose of numerical simulations, m_i and k_i are related to m_o and k_o as $m_i = m_o(1 + \delta m_i)$ and $k_i = k_o(1 + \delta k_i)$. The δm_i and the δk_i are randomly chosen using a uniform random number generator. The δm_i have a mean and standard deviation of $(-6.2\%, 28.0\%)$, and the δk_i have a mean and standard deviation of $(12.6\%, 25.4\%)$. To quantify the accuracy of the mass updating

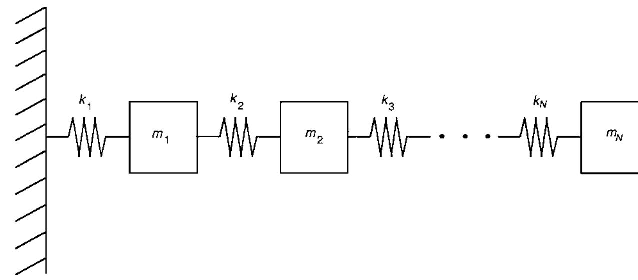

Fig. 1 Simple chain of coupled oscillators.

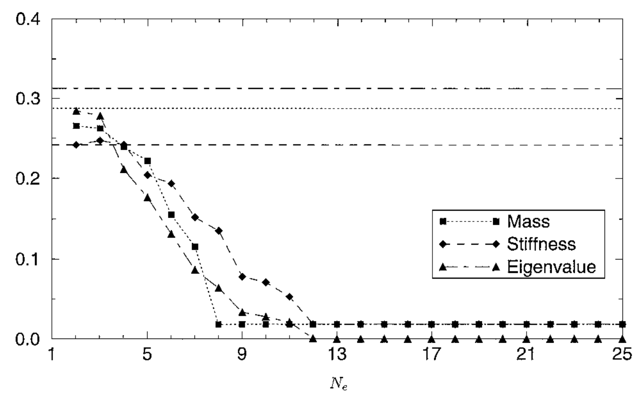

Fig. 2 Error parameters ϵ_m, ϵ_k, and ϵ_λ as a function of N_e; horizontal dotted lines represent the error parameters of the analytical model.

algorithm, the following error parameter for the updated masses is introduced:

$$\epsilon_m = \frac{|\boldsymbol{m}_{\text{update}} - \boldsymbol{m}_{\text{actual}}|}{|\boldsymbol{m}_{\text{actual}}|} \qquad (3)$$

where $|\boldsymbol{a}|$ represents the Euclidean norm of the vector $\boldsymbol{a}$. To illustrate the improvement of the updated masses over the initial analytical values, the following error parameter for the analytical masses is introduced:

$$(\epsilon_m)_o = \frac{|\boldsymbol{m}_{\text{analytical}} - \boldsymbol{m}_{\text{actual}}|}{|\boldsymbol{m}_{\text{actual}}|} \qquad (4)$$

Similar expressions can also be defined for the stiffness and the eigenvalue error parameters. An updated model is judged more accurate than the initial analytical model if $\epsilon_m < (\epsilon_m)_o$, $\epsilon_k < (\epsilon_k)_o$, and $\epsilon_\lambda < (\epsilon_\lambda)_o$. The smaller the error parameters are, the better the updated model is.

Figure 2 shows the variations of ϵ_m, ϵ_k, and ϵ_λ as a function of N_e. Also shown are the corresponding $(\epsilon_m)_o$, $(\epsilon_k)_o$, and $(\epsilon_\lambda)_o$, which are independent of N_e. For $N_e \geq 4$, the error parameters decrease as N_e increases. The results are consistent with physical intuition: the larger the knowledge space or the more information that can be gathered about the physical system, the better the updated model becomes. Note also that, for $N_e \geq 4$, $\epsilon_m > (\epsilon_m)_o$, $\epsilon_k > (\epsilon_k)_o$, and $\epsilon_\lambda > (\epsilon_\lambda)_o$. Thus, for the chosen set of system parameters, as long as $N_e \geq 4$, the proposed updating algorithms return a better updated model than the initial analytical model. In addition, observe that, for $N_e \geq 12$, $\epsilon_m \approx \epsilon_k \approx 1.8\%$ and $\epsilon_\lambda \approx 0$. Thus, there exists a critical N_e beyond which additional information does not lead to any improvements in the adjusted model.

Unlike the perturbation model updating approach introduced in Ref. 2, the proposed updating algorithms, which assume the connectivity information to be correct, are very forgiving when it comes to deviations of the analytical system from the physical structure. Specifically, the proposed updating schemes return accurate updated system matrices even when the initial analytical and the actual mass and stiffness parameters vary substantially from one another. Baruch,[4] in a recent Note, however, proved that even full modal data are insufficient for the identification of both the mass and stiffness matrices. He showed that the same modes of vibration can be obtained for an infinite number of different pairs of stiffness and mass matrices. Thus, the proposed schemes outlined in this Note will return updated system matrices that are correct only up to an arbitrary constant because both $[K]x = \lambda[M]x$ and $\alpha[K]x = \lambda\alpha[M]x$ (where α is an arbitrary positive constant) share the same eigensolutions. The results of Fig. 2 clearly support Baruch's contention. For $N_e \geq 12$, whereas ϵ_m and ϵ_k remain finite, $\epsilon_\lambda \approx 0$. To obtain the unique system matrices, we add known masses to the actual and the updated structures and compare the resulting generalized eigenvalue problems associated with these mass-modified systems. By algebraically manipulating the matrix equations, we can readily recover the appropriate α so that the updated matrices can be rendered unique.

For definiteness, assume $[\mathcal{M}]$ and $[\mathcal{K}]$ are the initial updated system matrices (that have already been found), and $\alpha[\mathcal{M}]$ and $\alpha[\mathcal{K}]$ correspond to the actual system matrices $[M]$ and $[K]$, respectively, where α is an unknown. To determine α, we add a known mass matrix $[\mathcal{M}_a]$ to both the actual and the analytical systems. Thus, the jth analytical and the jth actual modes of vibration must satisfy the following generalized eigenvalue problems:

$$[\mathcal{K}]\boldsymbol{x}'_j = \lambda'_j\{[\mathcal{M}] + [\mathcal{M}_a]\}\boldsymbol{x}'_j \tag{5}$$

$$\alpha[\mathcal{K}]\boldsymbol{x}''_j = \lambda''_j\{\alpha[\mathcal{M}] + [\mathcal{M}_a]\}\boldsymbol{x}''_j \tag{6}$$

where $(\lambda'_j, \boldsymbol{x}'_j)$ can be computed and $(\lambda''_j, \boldsymbol{x}''_j)$ can be measured. If $\alpha = 1$ (if the updated and the actual system matrices are identical), then $(\lambda'_j, \boldsymbol{x}'_j) = (\lambda''_j, \boldsymbol{x}''_j)$; otherwise $(\lambda'_j, \boldsymbol{x}'_j) \neq (\lambda''_j, \boldsymbol{x}''_j)$. Premultiplying Eq. (5) by $\boldsymbol{x}''^T_j$ and Eq. (6) by $\boldsymbol{x}'^T_j$ yields

$$\boldsymbol{x}''^T_j[\mathcal{K}]\boldsymbol{x}'_j = \lambda'_j\boldsymbol{x}''^T_j\{[\mathcal{M}] + [\mathcal{M}_a]\}\boldsymbol{x}'_j \tag{7}$$

$$\boldsymbol{x}'^T_j\alpha[\mathcal{K}]\boldsymbol{x}''_j = \lambda''_j\boldsymbol{x}'^T_j\{\alpha[\mathcal{M}] + [\mathcal{M}_a]\}\boldsymbol{x}''_j \tag{8}$$

Taking the transpose of Eq. (8), assuming $[\mathcal{K}]$, $[\mathcal{M}]$, and $[\mathcal{M}_a]$ are symmetric, subtracting the resulting equation from Eq. (7), and finally rearranging, we obtain

$$\alpha\left\{\boldsymbol{x}''^T_j[\mathcal{K}]\boldsymbol{x}'_j - \lambda''_j\boldsymbol{x}''^T_j[\mathcal{M}]\boldsymbol{x}'_j\right\} = \boldsymbol{x}''^T_j[\mathcal{K}]\boldsymbol{x}'_j - \lambda'_j\boldsymbol{x}''^T_j\{[\mathcal{M}] + [\mathcal{M}_a]\}\boldsymbol{x}'_j + \lambda''_j\boldsymbol{x}''^T_j[\mathcal{M}_a]\boldsymbol{x}'_j \tag{9}$$

For any given mode of vibration, Eq. (9) can be used to solve for the arbitrary constant α. Once it is found, we can tune the intial updated matrices to render them unique. Using the proposed tuning procedure with $N_e = 15$, the scaling factor was found to be $\alpha = 1.018$. Multiplying the initial updated matrices by 1.018, we obtain the unique updated matrices that are nearly identical to the actual system mass and stiffness matrices.

Conclusion

New mass and stiffness updating algorithms are developed. The proposed updating schemes are based on enforcing the orthogonality properties of distinct measured mode shapes with respect to the system mass and stiffness matrices. Manipulating the correction matrices $[\delta M]$ and $[\delta K]$ into vector forms, the connectivity information can be easily implemented. The proposed updating algorithms do not require the measured mode shapes to be normalized a priori, can be applied even for large differences between the actual and the analytical system matrices, and preserve the physical configuration of the structure.

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Modal Response of Trapezoidal Wing Structures Using Second-Order Shape Sensitivities

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Introduction

THE modal response of wing structures is very important for assessing their dynamic response, including dynamic aeroelastic instabilities. Moreover, in a recent study[1] an efficient structural optimization approach was developed using structural modes to represent the static aeroelastic wing response (both displacement and stress).

Sensitivity techniques are frequently used in structural design practices for seeking the optimal solutions near a baseline design.[2,3] The design parameters for wing structure include sizing-type variables (skin thickness, spar or rib sectional area, etc.), shape variables (the plan surface dimensions and ratios), and topological variables (total spar or rib number, wing topology arrangements, etc.). Sensitivities to the shape variables are extremely important because of the nonlinear dependence of stiffness and mass terms on the shape design variables as compared to the linear dependence on the sizing-type design variables.

Kapania et al.[4] and others[5–8] have obtained the first-order shape sensitivities of the modal response, divergence and flutter speed, and divergence dynamic pressure of laminated, box-wing, or general trapezoidal built-up wings using various approaches of determining the response sensitivities. In this Note, the modal response of general trapezoidal wing structures is approximated using shape sensitivities up to the second order. Also, different approaches of computing the derivatives are investigated.

Shape Sensitivities

For a trapezoidal wing, there are four major independent shape variables: 1) sweep angle Λ, 2) aspect ratio α, 3) taper ratio τ, and 4) plan area A. All of the other dimensions of the wing plate configuration can be calculated using these parameters, such as

$$s = \sqrt{\alpha A}, \qquad a = 2\tau s/\alpha(1+\tau), \qquad b = 2s/\alpha(1+\tau) \tag{1}$$

where s is the length of semispan and a and b are the chord length at wing tip and root, respectively, as shown in Fig. 1.

The sensitivities for the design parameters at a baseline design point indicate trends of variation of the design near the baseline point if the parameters are perturbed. Usually, only the first-order derivatives are used. For more accurate results, second-order derivatives can be used:

$$f(x^1, x^2, \ldots, x^n) \cong f\left(x_0^1, x_0^2, \ldots, x_0^n\right) + \sum_{i=1}^{n}\left(x^i - x_0^i\right)\frac{\partial}{\partial x^i} f\left(x_0^1, x_0^2, \ldots, x_0^n\right) + \frac{1}{2}\left[\sum_{i=1}^{n}\left(x^i - x_0^i\right)\frac{\partial}{\partial x^i}\right]^2 f\left(x_0^1, x_0^2, \ldots, x_0^n\right) \tag{2}$$

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